

ODE Existence Theorems: Picard-Lindelof, Cauchy-Peano, and Caratheodory

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Differential equations are important in many types for many reasons. They may be as simple as quantifying the exponential growth of bacteria dividing or more complex like tracking disease spread through an SIR model. In theoretical math, geodesic flows and Jacobi fields rely on differential equations for their definition and understanding solutions to things like the Laplace and Cauchy-Riemann equations tell us about fascinating types of functions. In most cases, we want to be able to solve the differential equations, but there is no unified method of doing so (and so we have several classes devoted to how to solve various types of them). In this document, we explore some theorems that tell us that solutions can exist in special cases. In particular, we focus on ordinary differential equations (ODEs), which are single-variable differential equations. In general, they will take the form

$$(A) \begin{cases} x'(t) = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

There is a fascinating theory of weak solutions to differential equations that we will not consider here. We will consider a solution to be a function $x : I \rightarrow \mathbb{R}^n$ satisfying the above criteria (so $t_0 \in I$ and x must be differentiable). We may want to specify what I is, but sometimes we cannot. For example,

$$x' = \frac{1}{t}$$

has the general solution $x(t) = \ln(t) + C$, which cannot be extended to negative time values.

Further, recall that we only need one derivative because higher-order ODEs may be written as a system of first-order ODEs.

[Some Bad Cases]

Picard Iteration, Existence and Uniqueness

The first theorem we will explore relies on writing the differential equation as a functional equation so that the solution is a fixed point. Then the Contraction Mapping Theorem (also called the Banach Fixed Point Theorem) provides the existence of a solution. Recall that theorem: **Contraction Mapping Theorem** Let X be a complete metric space, and let $f : X \rightarrow X$ be such that $d(f(x), f(y)) \leq cd(x, y)$ for $c \in (0, 1)$ and all $x, y \in X$. Then, there exists a unique x^* so $f(x^*) = x^*$.

Proof: Let $x_0 \in X$ be any point. We generate a sequence $x_i = f(x_{i-1})$. Since f is a contraction, we have by the triangle inequality that

$$d(x_{n+k}, x_n) \leq c^n d(x_k, x_0) \leq c^n \left(\sum_{i=0}^{k-1} d(x_i, x_{i+1}) \right) \leq c^n \left(\sum_{i=0}^{k-1} c^i \right) d(x_0, x_1)$$

Since $|c| < 1$, $\sum_{i=0}^{\infty} c^i$ is bounded, and $c^n \rightarrow 0$ as $n \rightarrow \infty$. Hence, the sequence $\{x_n\}$ is Cauchy and has some limit x^* . Since f is continuous, we must have $x^* = f(x^*)$ by the definition of the sequence. If y^* is any other

point so $f(y^*) = y^*$, then $d(f(y^*), f(x^*)) = d(x^*, y^*)$, contradicting the contraction property of f . Hence, x^* is unique in this property. *Q.E.D*

A key point is that we may start with *any* point in the metric space and iterate the function to reach the fixed point. Let us try to apply this to an ODE. We start by assuming f is continuous on some closed rectangle $D \subset \mathbb{R} \times \mathbb{R}^n$ so that we don't get the issue we say above. For use later, let $D = J \times A$ where J is a closed interval and A a closed rectangle that contain t_0 and x_0 in their interior (in some ball inside each of them). This continuity assumption isn't quite strong enough. Consider the example

$$x'(t) = tx^2$$

Solving produces a solution $\frac{2}{t^2+C} = x$. Thus, for some initial conditions, we may fail to have a solution. We restrict further to forcing f to be Lipschitz continuous in x with a constant uniform in t , i.e. $|f(t, x) - f(t, y)| \leq M|x - y|$ for some $M > 0$ that doesn't depend on t . Lastly, we pick our metric space. If f is continuous, we should expect a solution x to be C^1 . This is indeed a metric space under the norm $\|f\|_{C^1} = \|f\|_{\infty} + \|f'\|_{\infty}$, but having this derivative in the norm will add complexity. Further, the derivative in the ODE may take C^1 functions outside of C^1 , so to fit the above theorem we change the differential equation to the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds$$

and define $F(y)(t) = x_0 + \int_{t_0}^t f(s, y(s))ds$ a map from $C^0(J) \rightarrow C^1(J) \subset C^0(J)$. We then decide on continuous functions being our metric space, but we also need to keep the functions within the set A . In fact, what we do is restrict to $I_{\epsilon}(t_0) \times B_{\delta}(x_0) = [t_0 - \epsilon, t_0 + \epsilon] \times \{x \in \mathbb{R}^n : |x - x_0| \leq \delta\}$ and look at applying F to $C^0(I_{\epsilon}(t_0), B_{\delta}(x_0))$ (where we pick ϵ and δ soon). First, we need to show that F should map this set to itself, which amounts to checking whether $F(y)$ grows too large:

$$|F(y(t)) - x_0| = \left| \int_{t_0}^t f(s, y(s))ds \right| \leq \epsilon \|f\|_{C^0(I_{\epsilon}(t_0) \times B_{\delta}(x_0))}$$

First, if we pick ϵ and δ small enough, we may replace the norm of f by $\|f\|_{C^0(D)}$. We then just need $\delta \geq \epsilon \|f\|_{C^0(D)}$.

We next want F to be a contraction, and this is where we leverage the weird Lipschitz assumption:

$$\|F(y) - F(x)\| \leq \sup_{t \in J} \int_{t_0}^t |f(s, y(s)) - f(s, x(s))|ds \leq \epsilon M \|y - x\|_{C^0(I_{\epsilon}(t_0))}$$

so that F is a contraction if $\epsilon M < 1$. Hence, let δ be small enough so that $B_{\delta}(x_0) \subset A$ and let $\epsilon \leq \min\{\frac{1}{2M}, \delta/\|f\|_{C^0(D)}\}$ be small enough so $I_{\epsilon}(t_0) \subset J$. We now have a contraction on $C^0(I_{\epsilon}(t_0), B_{\delta}(x_0))$, which gives a fixed point $x(t)$! Since $x(t)$ satisfies the integral equation, it is C^1 , solves the ODE (A), and is unique by virtue of being the fixed point.

We have proved the following theorem

Picard-Lindelöf Theorem

Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be a closed rectangle with (t_0, x_0) contained in some ball $B \subset D$. If $f : D \rightarrow \mathbb{R}$ is continuous in t (for all y) and Lipschitz continuous in y (with a Lipschitz constant independent from t), then the ODE (A) has a unique local solution near t_0 [2][1].

The contraction F that we constructed is called the Picard operator, and gives us one method to solve the ODE. If we pick an arbitrary function $y \in C^0(I_{\epsilon}(t_0), B_{\delta}(x_0))$, the sequence $\{F^n(y)\}$ converges to our solution. Computing this sequence is often called Picard iteration, though determining a limit can be tricky at times.

While this solution only provides a local solution, if D is the whole space $\mathbb{R} \times \mathbb{R}^n$, we may piece together local solutions to see global solutions. Often, this is easy to do when solving but hard to work with in generality.

Peano: Weaker Conditions for Existence

The Picard-Lindelöf theorem gives existence of solutions for a broad class of ODEs, but a Lipschitz assumption is quite strong. Here, we display how to weaken that at the cost of uniqueness. This also relies on two big theorems— the Stone-Weierstrass Theorem and the Arzela-Ascoli Theorem (I have another document on these)— and a good bit more effort.

We only assume that $f : D \rightarrow \mathbb{R}$ is continuous on D . Let $M = \sup_D |f|$. Then, the Stone-Weierstrass Theorem guarantees a sequence of Lipschitz functions $\{f_n\}$ converges to f . By removing some terms of the sequence, we may assume $|f_n| \leq 2M$ for all n .

If we pick the ϵ and δ from the previous proof using the rectangle D and the constant $2C$, they don't depend on f . Let us now reduce to focusing on the rectangle picked for the Picard-Lindelöf Theorem under these constants. We may define F_n using f_n and obtain a sequence $\{y_{n,k}\}$ so $y_n = \lim_{k \rightarrow \infty} y_{n,k}$ is the unique fixed point of F_k . Furthermore,

$$|y_{n,k}(t') - y_{n,k}(t)| \leq \int_t^{t'} |f_n(s, y_{n,k-1}(s))| ds \leq 2M|t' - t|$$

so that these functions are equicontinuous, whereby Arzela-Ascoli dictates that $\{y_{n,k}\}$ is relatively compact. Since the y_n are limit points of this set, they themselves are relatively compact and must have some limit point y and a subsequence $y_{n_m} \rightarrow y$. Since

$$y_{n_m}(t) = x_0 + \int_{t_0}^t f_{n_m}(s, y_{n_m}(s)) ds$$

taking the limit on both sides (and noting that the f_n are equicontinuous) shows that $y(t) = x_0 + \int_{t_0}^t f(s, y(s)) ds$. Finally, we have a solution! This amounts to the following theorem.

Cauchy-Peano Theorem

Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open subset, where $f : D \rightarrow \mathbb{R}^n$ is a continuous function. For every $(t_0, x_0) \in D$, the ODE (A) has a local solution $x : I \rightarrow \mathbb{R}^n$ [3].

We can upgrade this theorem in multiple ways. First, we can find even weaker assumptions using measure theory. This is useful in some contexts, but we will not treat it here. If you are interested, the main result is called Caratheodoty's existence theorem. Second, we may notice that the restrictions on ϵ and δ depend on how close the initial point is to the edge of the rectangle, and so depend on the initial conditions, but the restrictions based on f are uniform in the initial conditions. This allows us to piece together local solutions as mentioned above. An example of a precise bound doing this is the following.

Quantifying the Local Interval

If f is continuous in a rectangle

$$R = \{(t, x) : |t - t_0| \leq \alpha, |x - x_0| \leq \beta\}$$

and bounded by a value M , then the ODE (A) has a solution $x(t)$ for $|t - t_0| \leq \min\{\alpha, \beta/M\}$.

Ending Note

These are two introductory theorems into the realm of ODE existence and uniqueness, but there are certainly many, many more. In standard practice, the above two will usually suffice, but in the special occasions that come up in research, these assumptions can be too strong or not tell use about what existence implies. Offhand, I know of at least one book fully devoted to the study of such criteria, *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations* by R.P. Agarwal and V. Lakshmikantham (<https://doi.org/10.1142/1988>, Published 1993). This is a good place to start if you're looking for such criteria, like Okamura's Uniqueness Theorem, but even more specific and recent results have come out since this was published.

References

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